

Random planar geometry

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Cambridge

April 6, 2019

Overview

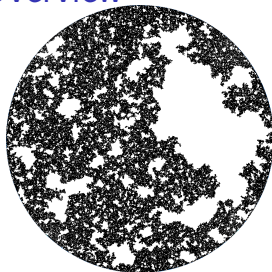
Random two-dimensional geometry is an area of **probability theory** which studies random **planar curves**, **shapes**, and **surfaces**.

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Motivated by questions from **statistical mechanics**

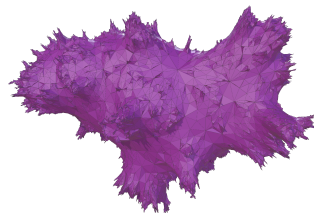
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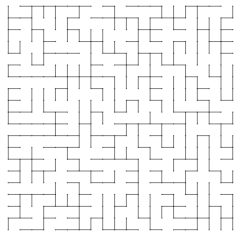
Percolation



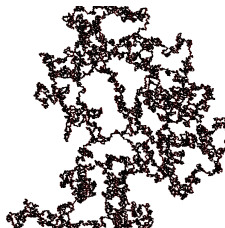
Brownian motion



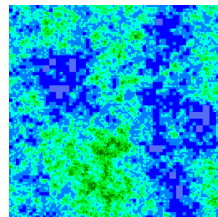
Random planar maps



Uniform spanning tree



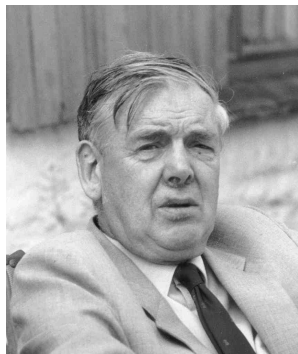
Schramm-Loewner
evolution



Liouville quantum gravity

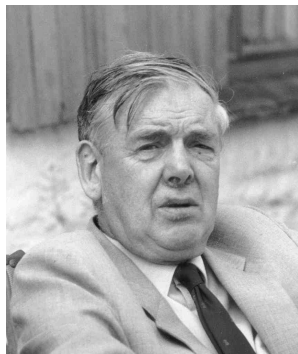
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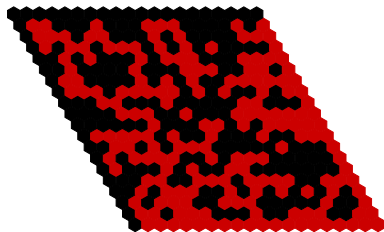
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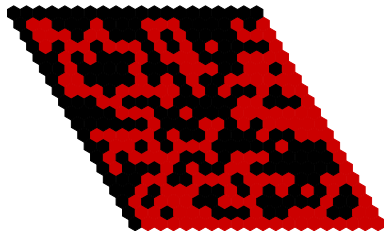
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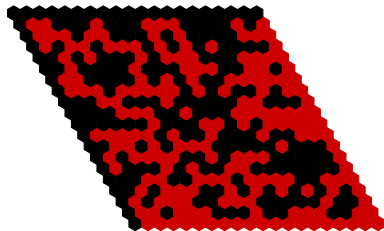
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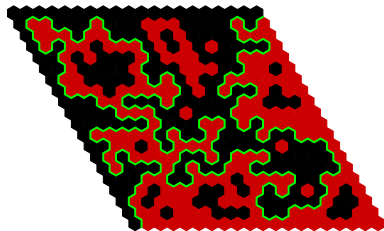
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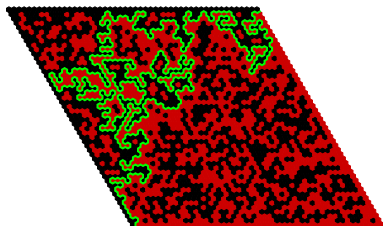
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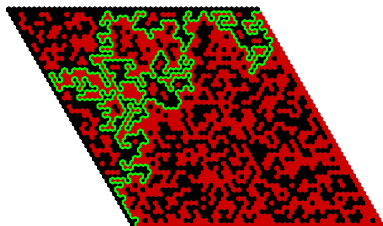
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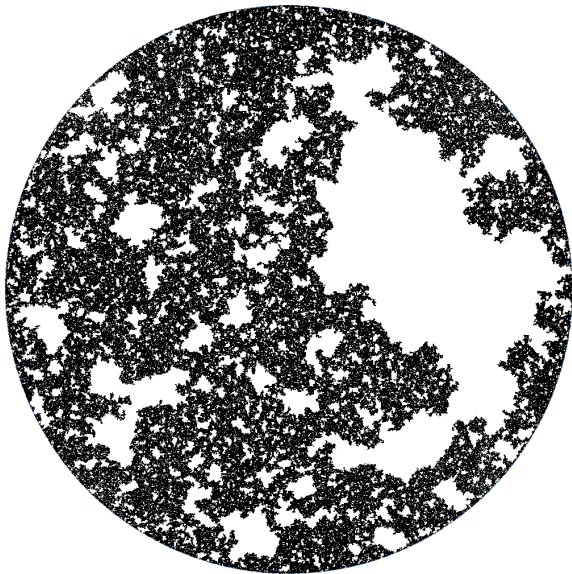
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- ▶ How does one mathematically describe the limit of the interface as the mesh size $\rightarrow 0$?





Percolation on a 1000×1000 box, conformally mapped to $\mathbf{D}$. Shown are the clusters which touch the boundary.

Two-dimensional Brownian motion



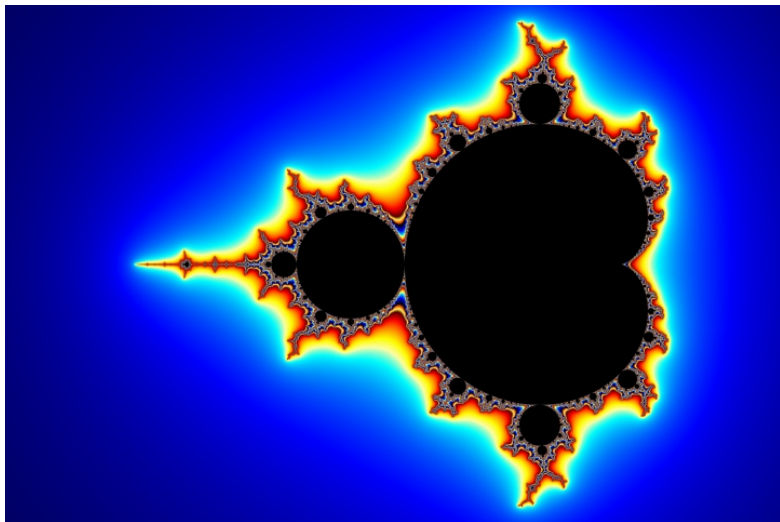
Two-dimensional Brownian motion

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Benoit Mandelbrot (1924-2010)

Two-dimensional Brownian motion



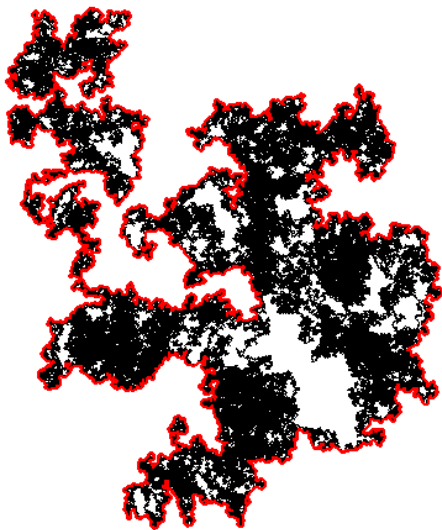
Mandelbrot set

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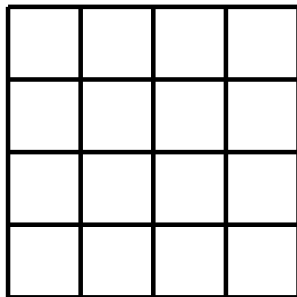
Two-dimensional Brownian motion



Can one mathematically analyze the outer boundary of Brownian motion?

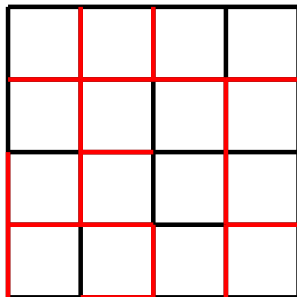
Uniform spanning tree

- Suppose that $G = (V, E)$ is a graph



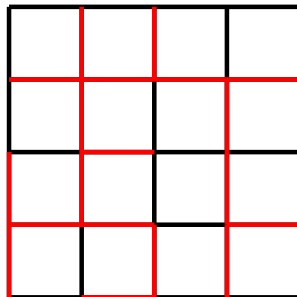
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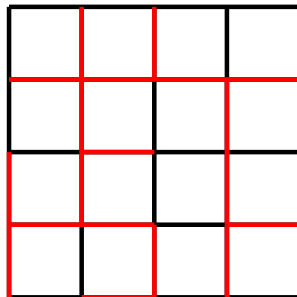
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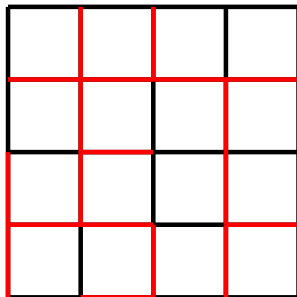
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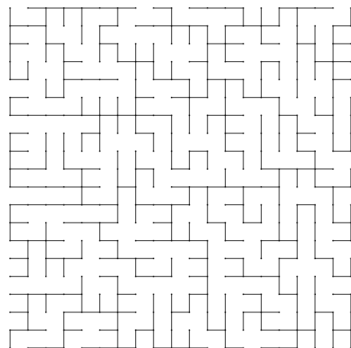
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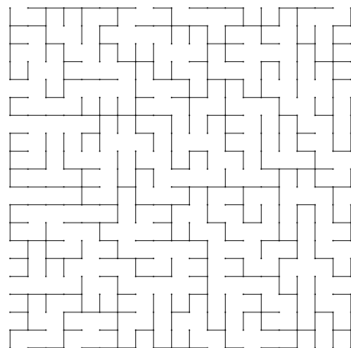
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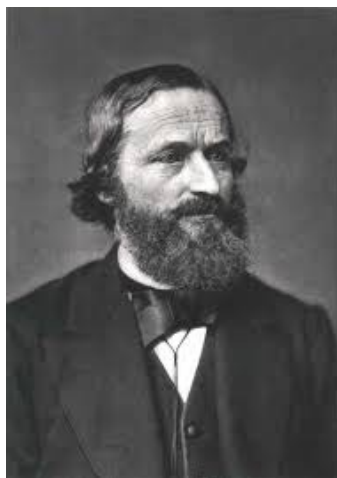
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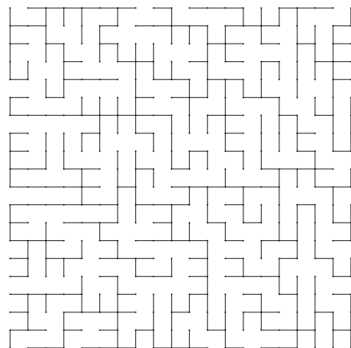
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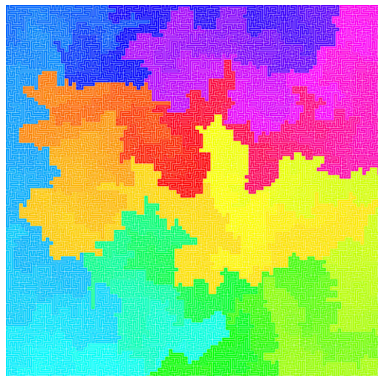
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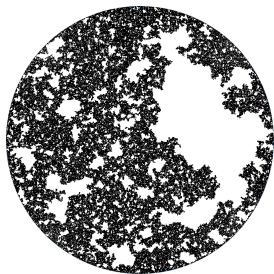


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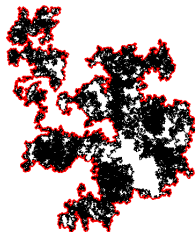
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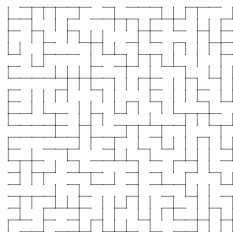
Recap



Percolation

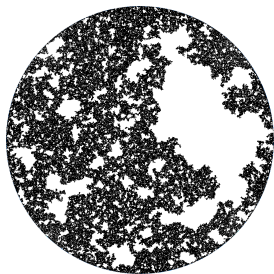


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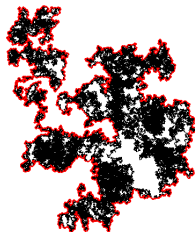


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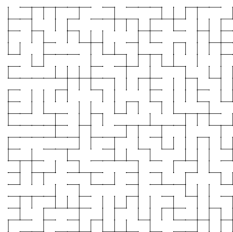
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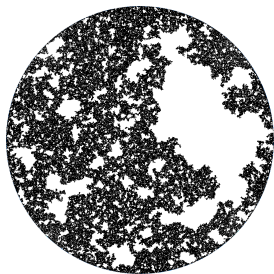
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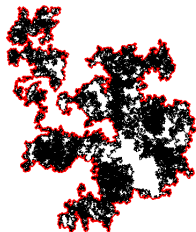
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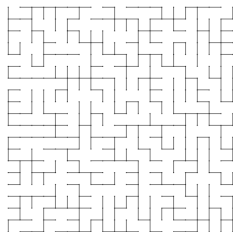
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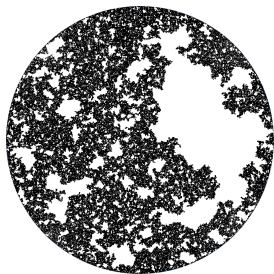
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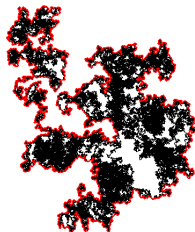
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- ▶ Models (and others) have been around for many decades, studied by mathematicians and physicists
- ▶ Tremendous progress in the last 20 years, much of it starting with an idea of **Oded Schramm** combining ideas from **complex analysis**, **geometry**, and **probability**

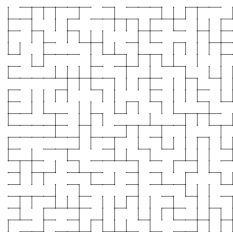
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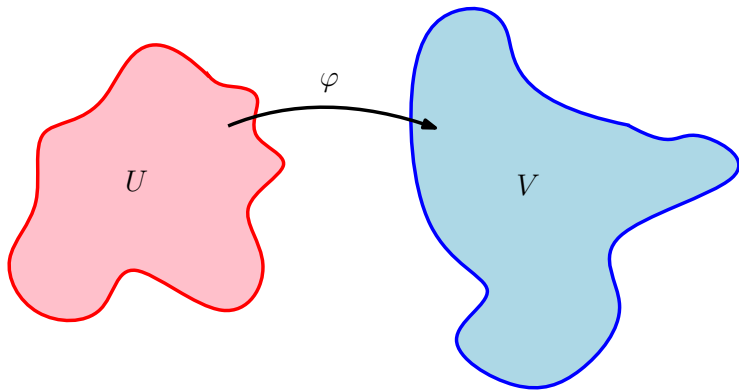


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- ▶ In a nutshell, Schramm discovered a tractable way of describing the random curves which appear in these models in a unified manner

Riemann mapping theorem

Suppose that U, V are simply connected domains in $\mathbf{C}$. Then there exists a conformal map $\varphi: U \rightarrow V$.



In fact, there is exactly a three-parameter family of conformal maps.

Loewner evolution

- ▶ Mathematical tool invented by Charles Loewner in the 1920s to encode curves in $\mathbf{C}$



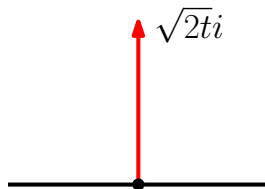
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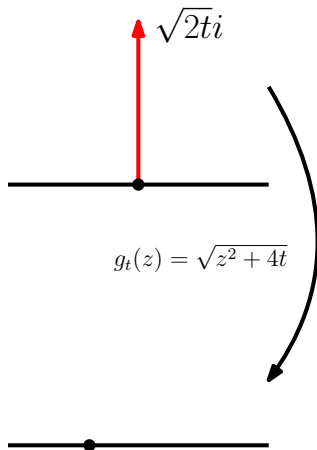
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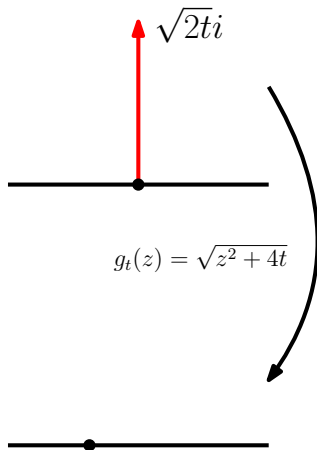
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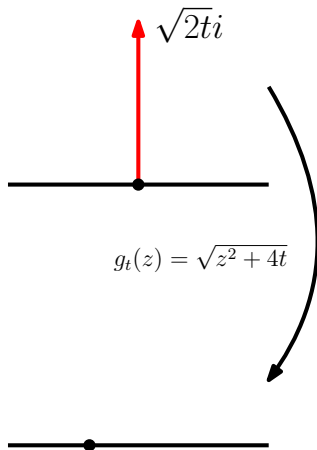
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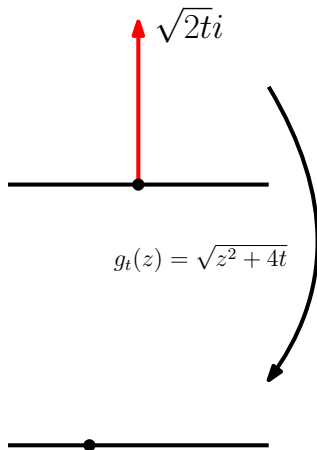
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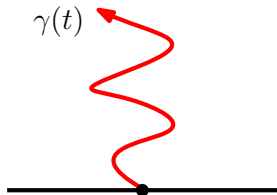
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- ▶ This is called the **Loewner evolution** for γ



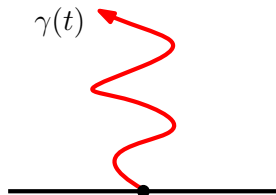
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Loewner evolution

- ▶ Suppose that γ is a non-crossing curve in $\mathbf{H}$ from 0 to ∞
- ▶ Let g_t be the unique conformal map $\mathbf{H} \setminus \gamma([0, t])$ to $\mathbf{H}$ with $g_t(z) - z \rightarrow 0$ as $z \rightarrow \infty$

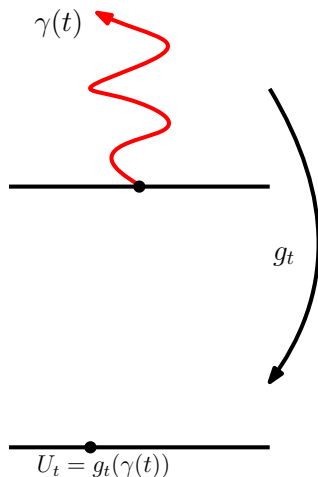


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- ▶ **Theorem:** The maps g_t satisfy the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z$$

where U is a continuous $\mathbf{R}$ -valued function



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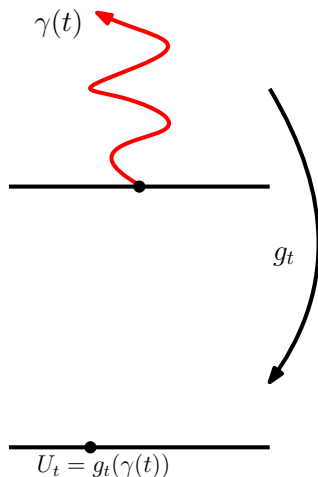
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- ▶ Curves in $\mathbf{H}$ can be encoded in terms of $\mathbf{R}$ -valued continuous functions



Oded Schramm

- PhD in complex analysis and geometry under Thurston



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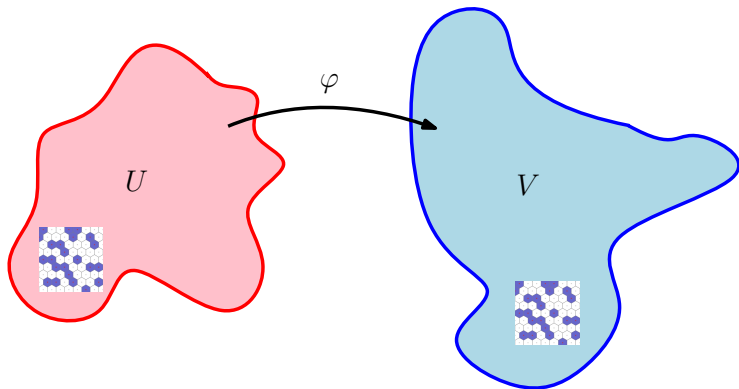
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- ▶ **Idea:** Use Loewner evolution to study random curves which come up in random planar geometry
 - ▶ percolation paths
 - ▶ uniform spanning tree branches
 - ▶ etc...



Conformal invariance

Physics idea: many models in two-dimensional random geometry should be **conformally invariant**



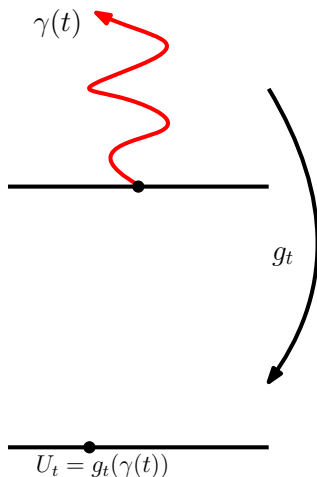
If $\varphi: U \rightarrow V$ is a conformal map, then the image under φ of percolation interfaces in U look like percolation interfaces in V .

Schramm-Loewner evolution

- ▶ Suppose that γ is a non-crossing curve in $\mathbf{H}$ from 0 to ∞
- ▶ Let g_t be the unique conformal map $\mathbf{H} \setminus \gamma([0, t])$ to $\mathbf{H}$ with $g_t(z) - z \rightarrow 0$ as $z \rightarrow \infty$
- ▶ **Theorem:** The maps g_t satisfy the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z$$

where U is a continuous $\mathbf{R}$ -valued functions



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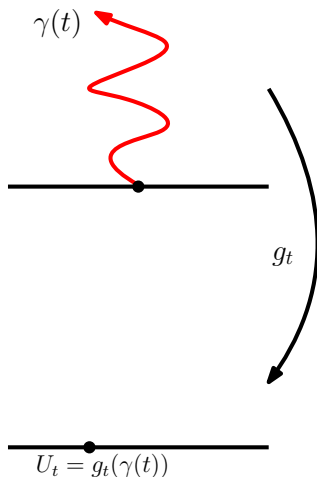
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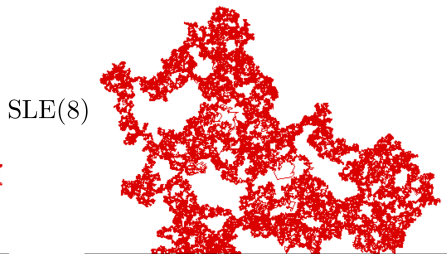
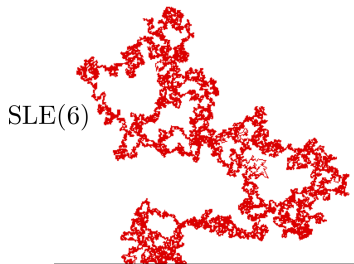
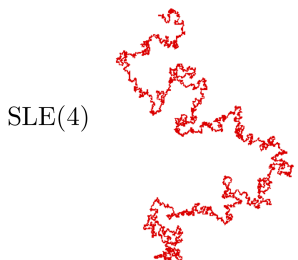
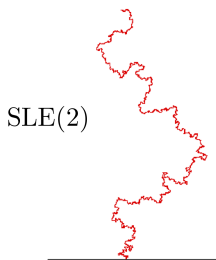
- ▶ **Theorem:** The maps g_t satisfy the ODE

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where U is a continuous $\mathbf{R}$ -valued functions

- ▶ **Schramm's observation:** if γ is a random curve which comes from a conformally invariant model (e.g., percolation) then U_t must be a multiple $\sqrt{\kappa}$ of a Brownian motion.

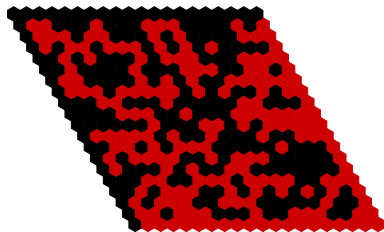




Simulations due to Tom Kennedy.

Convergence of percolation to SLE_6

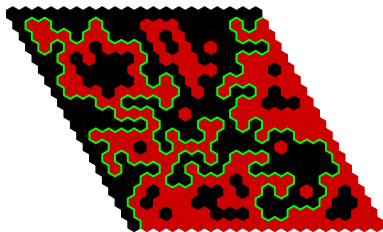
- **Smirnov:** The exploration path between open and closed sites in critical site percolation on the $\triangle$ -lattice converges to SLE_6 as the mesh size tends to 0



Stanislav Smirnov

Convergence of percolation to SLE_6

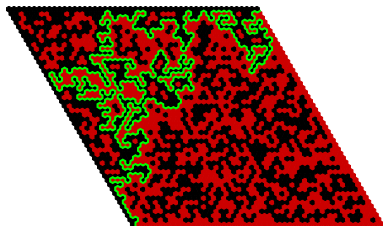
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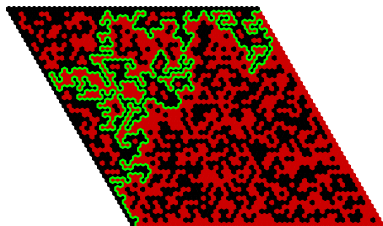
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Stanislav Smirnov

Convergence of percolation to SLE_6

- ▶ **Smirnov:** The exploration path between open and closed sites in critical site percolation on the $\triangle$ -lattice converges to SLE_6 as the mesh size tends to 0
- ▶ **Open problem:** prove the convergence of percolation to SLE_6 on any other planar lattice



Stanislav Smirnov

Self-avoiding walk (SAW)

- ▶ Invented by Flory in 1953



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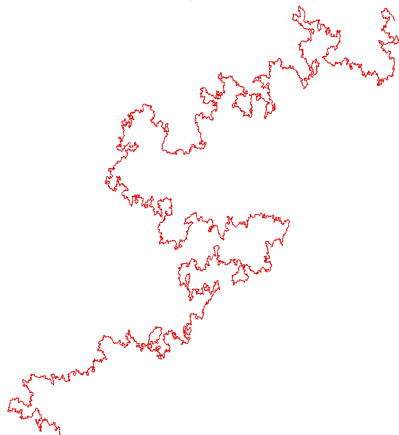
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- ▶ Uniform measure on **non-self-intersecting paths** starting from x in G of length n



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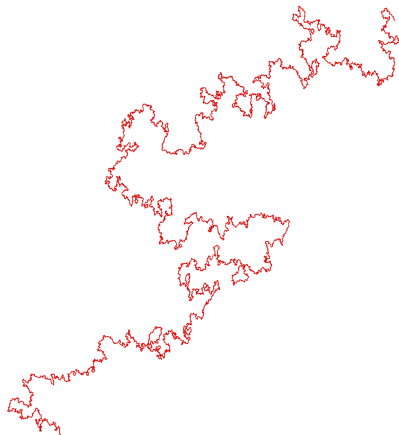
SAW in plane - 1,000,000 steps



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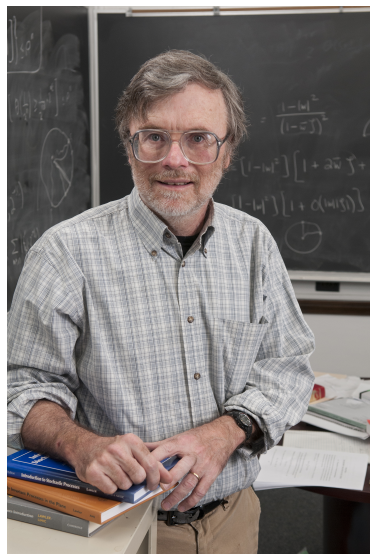
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SAW in plane - 1,000,000 steps



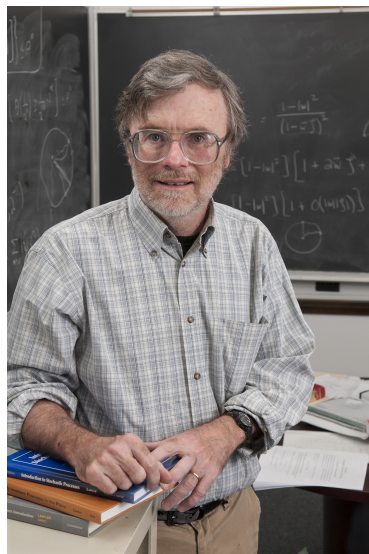
Loop-erased random walk and the uniform spanning tree

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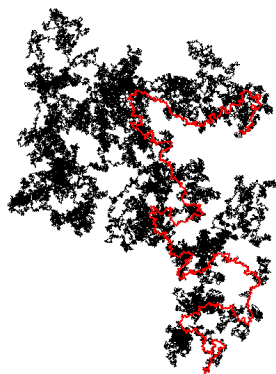
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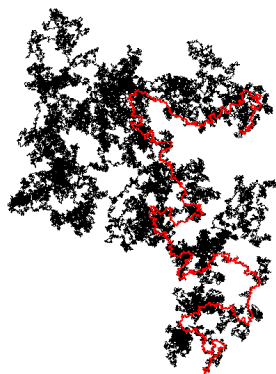
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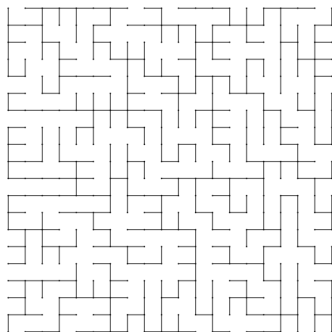
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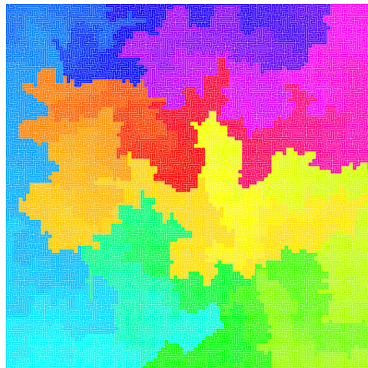
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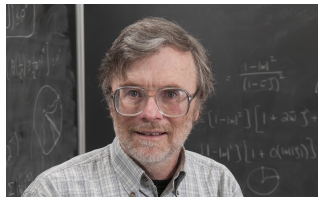
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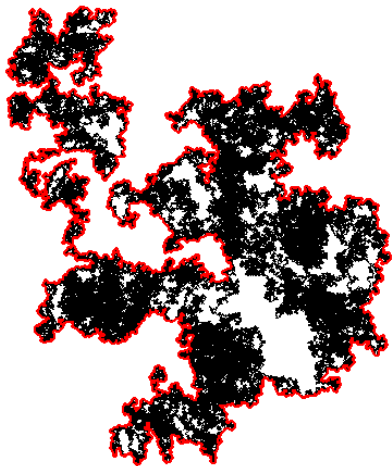
Outer boundary of Brownian motion



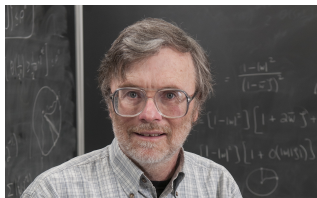
Lawler-Schramm-Werner: the outer boundary of Brownian motion is $\text{SLE}_{8/3}$



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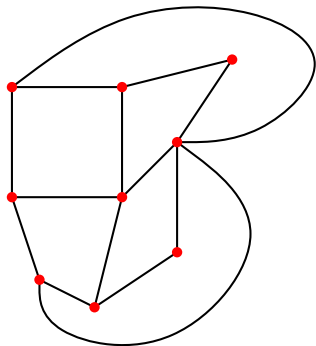


Random surfaces

- ▶ There has been tremendous progress in understanding **random curves** in $\mathbb{C}$
- ▶ In recent years, there has been a lot of work aimed at understanding **random surfaces**
 - ▶ **Motivation:** physicists starting back in the 1970s has a precise picture of how many critical models in two-dimensional statistical mechanics should behave at large scales.
 - ▶ It is easier to understand these models on a **random surface** than on a planar lattice.

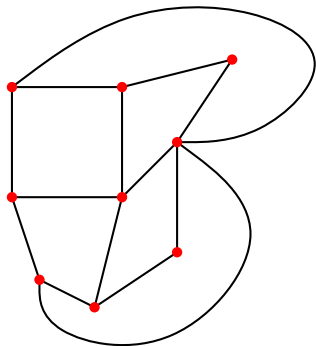
Random planar maps

- ▶ A **planar map** is a finite graph together with an embedding in the plane so that no edges cross

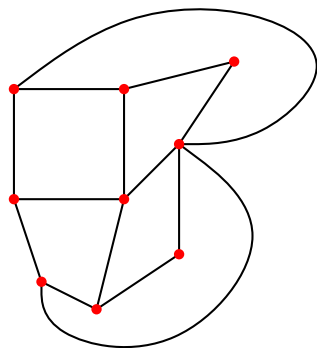


Random planar maps

- ▶ A **planar map** is a finite graph together with an embedding in the plane so that no edges cross
- ▶ Its **faces** are the connected components of the complement of its edges

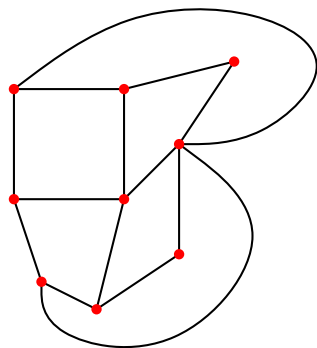


Random planar maps



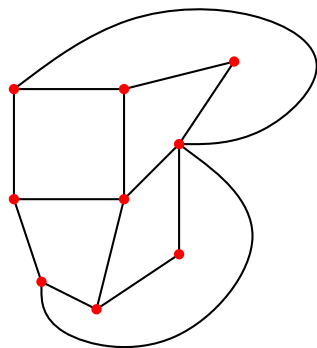
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Random planar maps



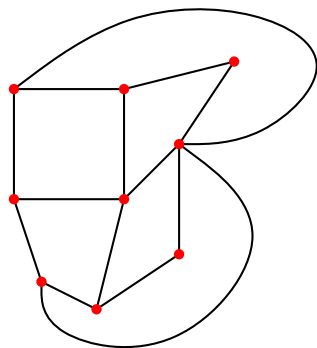
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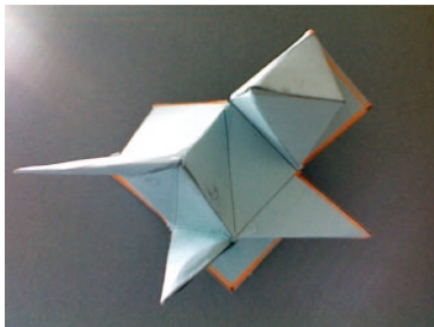
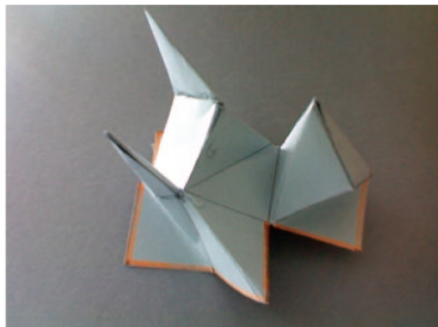


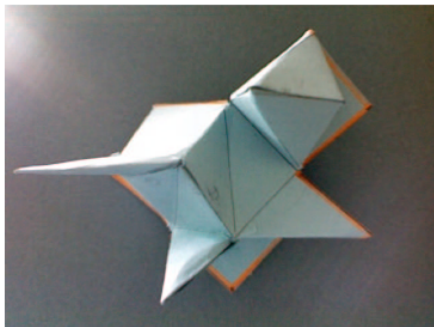
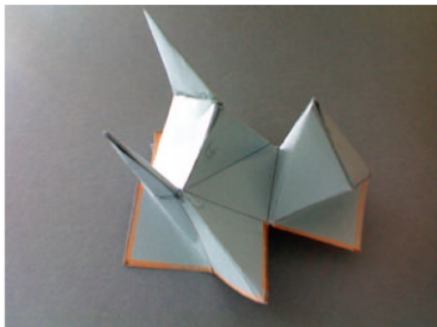
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Random planar maps

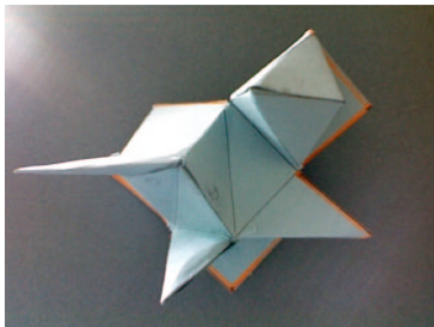
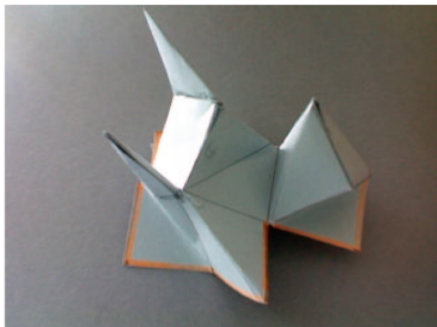


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- ▶ A quadrangulation corresponds to a **metric space** when equipped with the graph distance
- ▶ Interested in **uniformly random quadrangulations** with n faces — **random planar map** (RPM).
- ▶ First studied by Tutte in 1960s while working on the four color theorem
 - ▶ **Combinatorics**: enumeration formulas
 - ▶ **Physics**: statistical physics models: percolation, Ising, UST ...
 - ▶ **Probability**: “uniformly random surface,” Brownian surface





What is the structure of a typical quadrangulation when the number of faces is large?

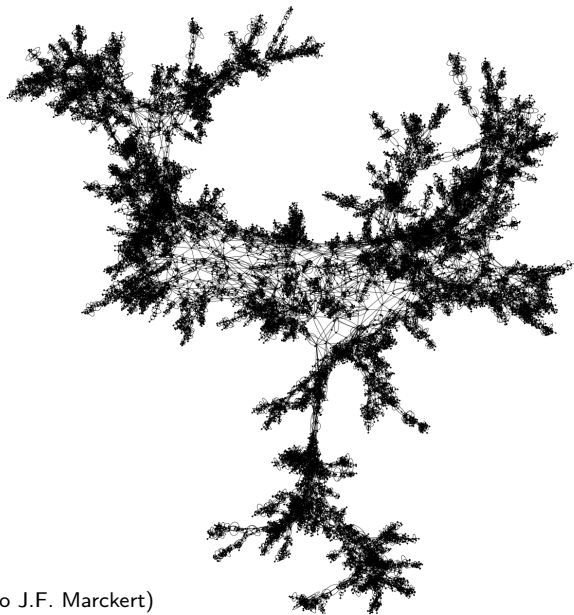


What is the structure of a typical quadrangulation when the number of faces is large?

How many are there? **Tutte:**

$$\frac{2 \times 3^n}{(n+1)(n+2)} \binom{2n}{n}.$$

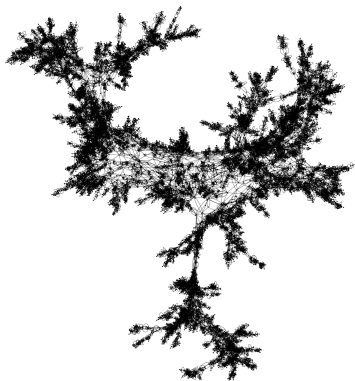
Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Structure of large random planar maps

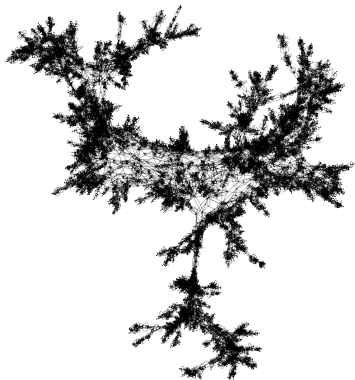
- RPM as a **metric space**. Is there a limit?



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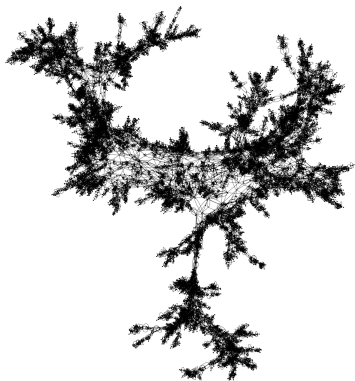
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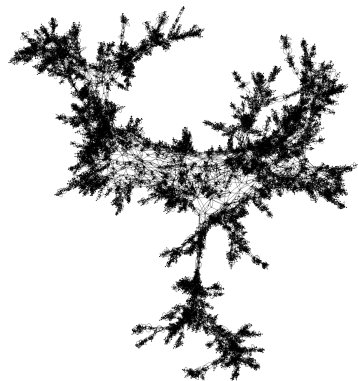
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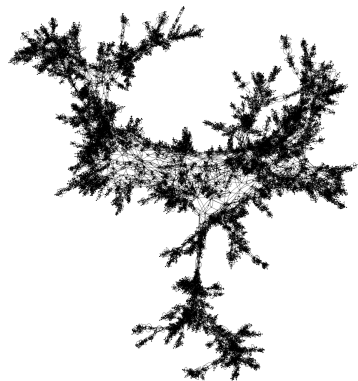
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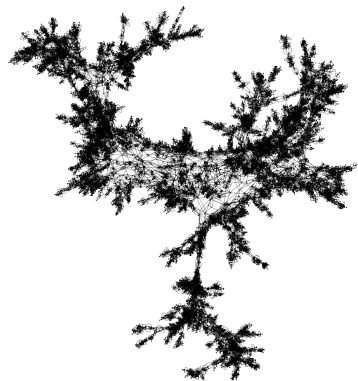
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Structure of large random planar maps



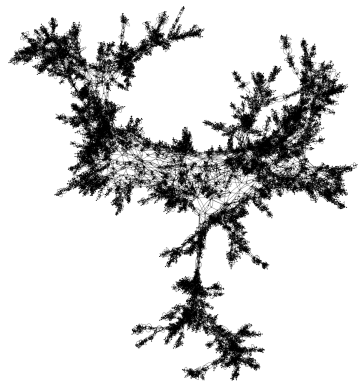
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Important tool: bijections which encode the surface using a gluing of a pair of trees

(Mullin, Schaeffer, Cori-Schaeffer-Vauquelin, Bouttier-Di Francesco-Guitter, Sheffield,...)

Structure of large random planar maps



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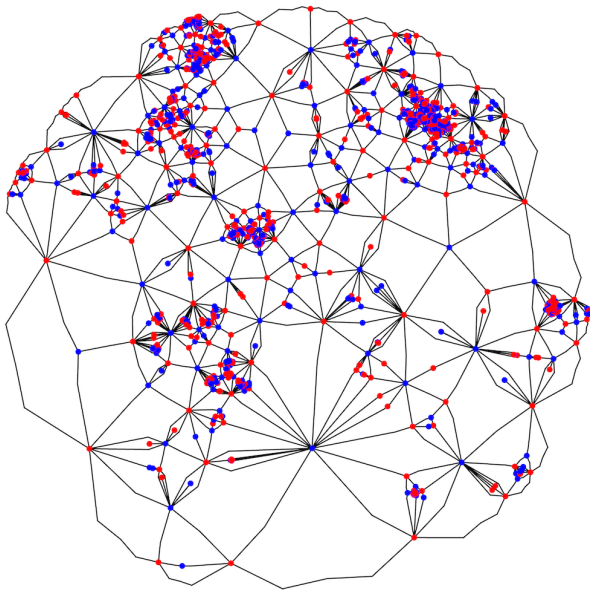
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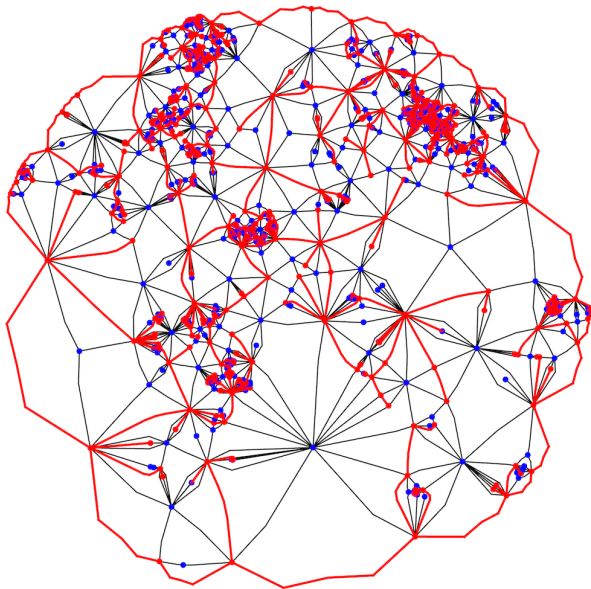
Brownian map also described in terms of trees (CRT)

Random quadrangulation



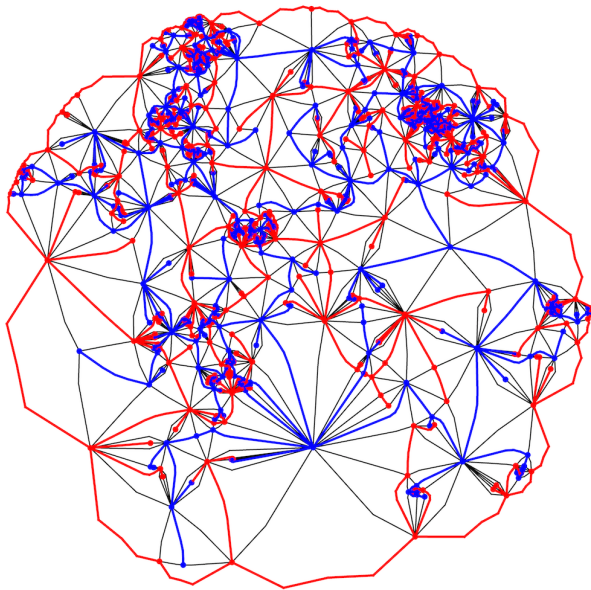
Sampled using H-C bijection.

Red tree



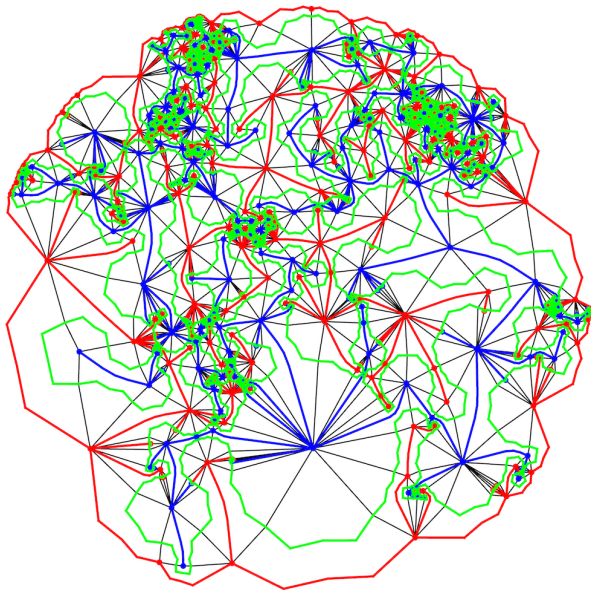
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Red and blue trees



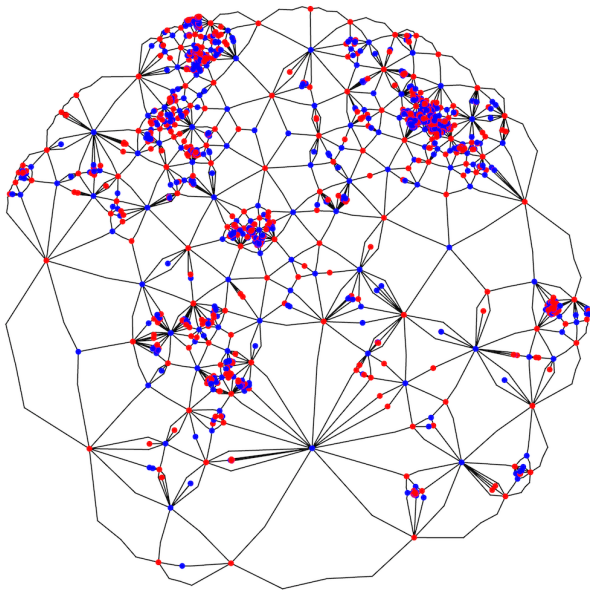
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Path snaking between the trees. Encodes the trees and how they are glued together.



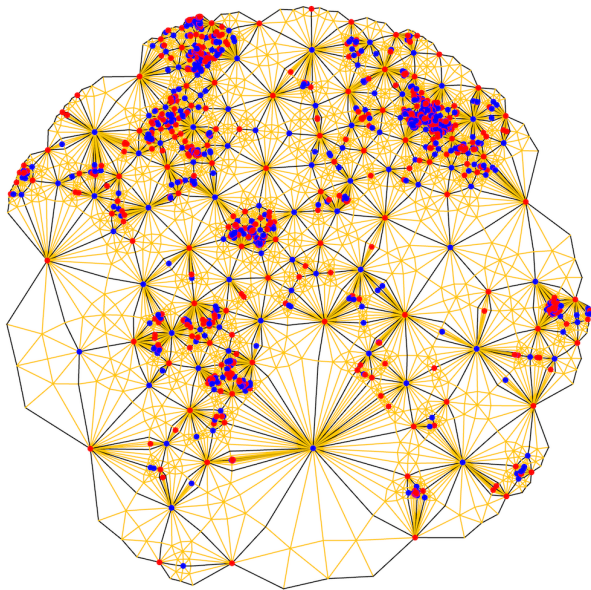
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How was the graph embedded into $\mathbf{R}^2$?



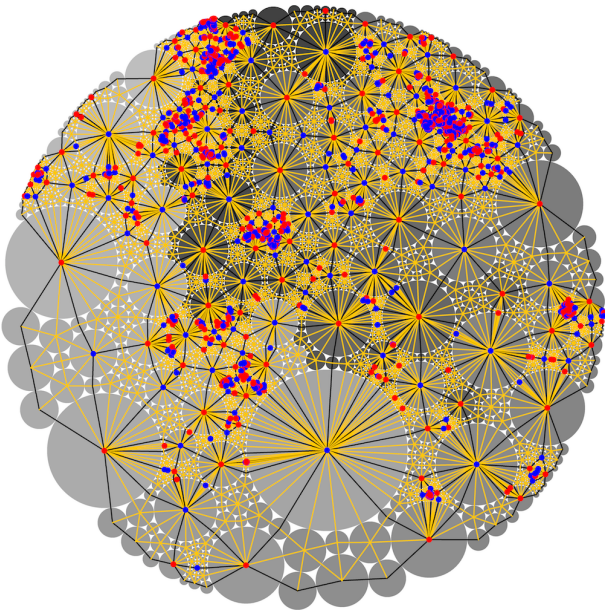
Sampled using H-C bijection.

Can subdivide each quadrilateral to obtain a triangulation without multiple edges.



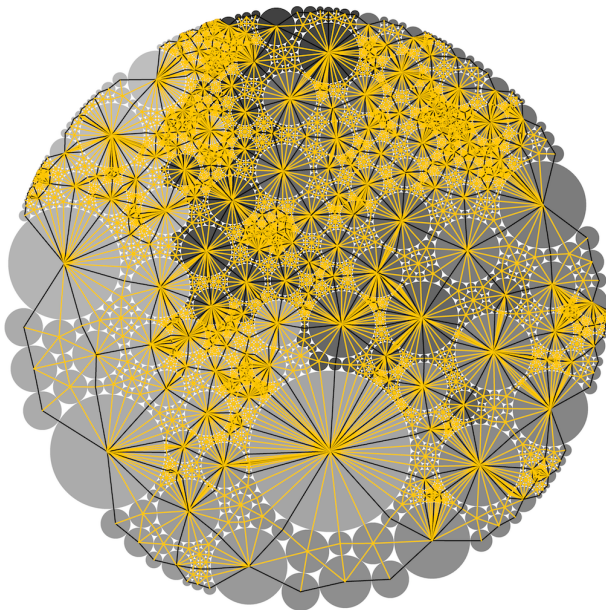
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Circle pack the resulting triangulation.



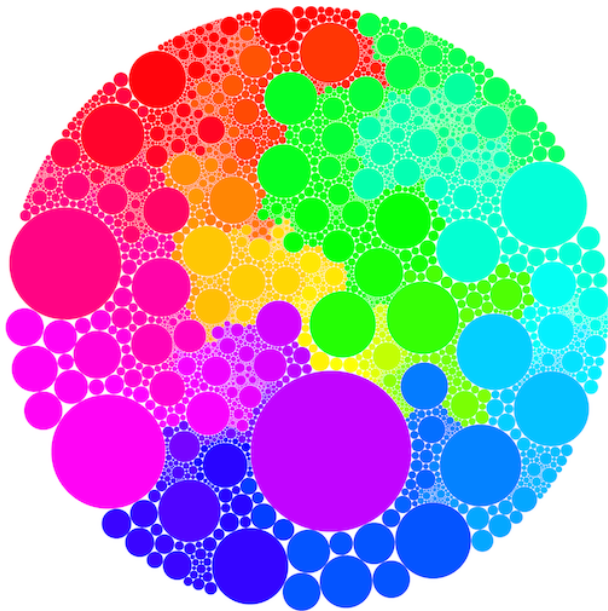
Sampled using H-C bijection. Packed with Stephenson's CirclePack.

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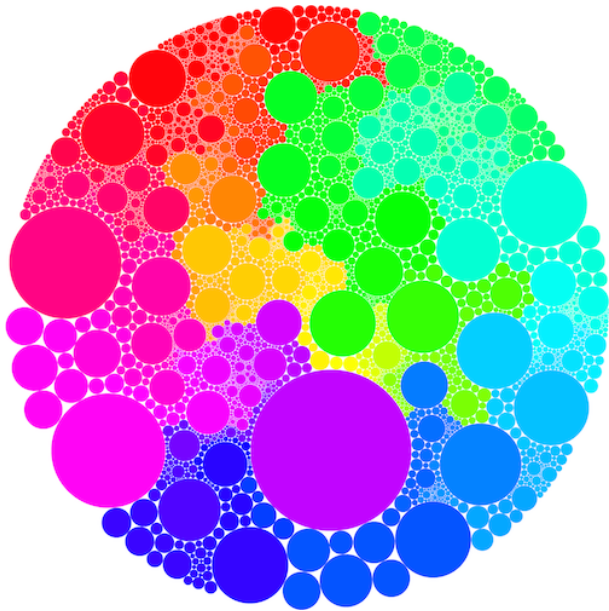
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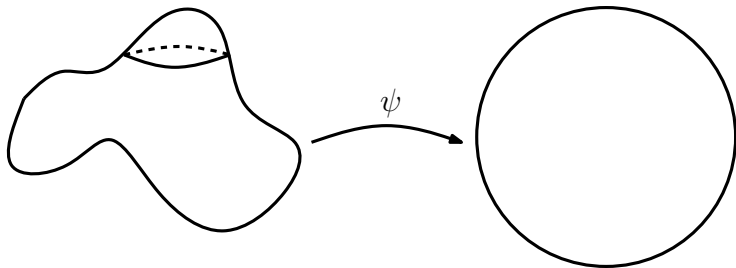
What is the “limit” of this embedding? Circle packings are related to conformal maps.



Sampled using H-C bijection. Packed with Stephenson's CirclePack.

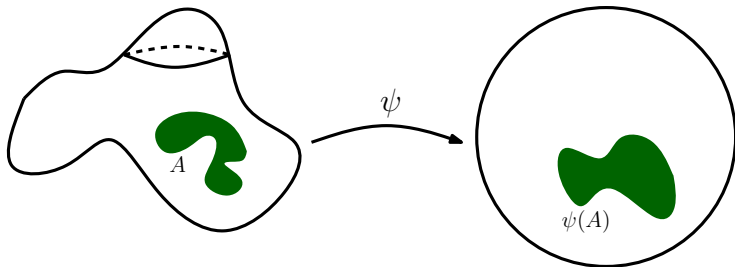
Picking a surface at random in the continuum

Uniformization theorem: every Riemannian surface homeomorphic to the unit disk $\mathbf{D}$ can be conformally mapped to the disk.



Picking a surface at random in the continuum

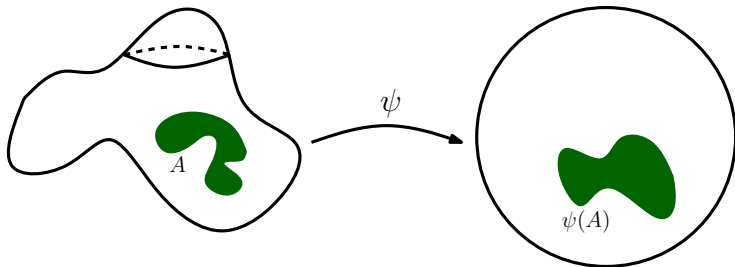
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Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}(dx^2 + dy^2)$ for some smooth function ρ where $dx^2 + dy^2$ is the Euclidean metric.

Picking a surface at random in the continuum

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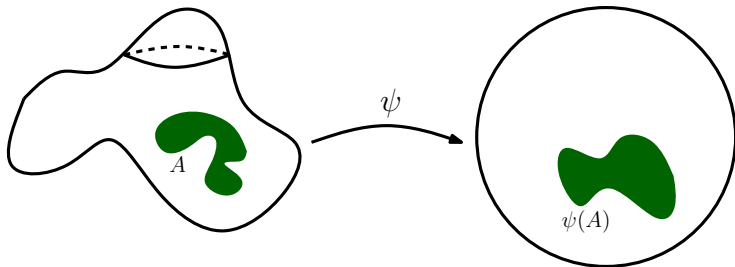
Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}(dx^2 + dy^2)$ for some smooth function ρ where $dx^2 + dy^2$ is the Euclidean metric.

$\Rightarrow$ Can parameterize the surfaces homeomorphic to $\mathbf{D}$ with smooth functions on $\mathbf{D}$.

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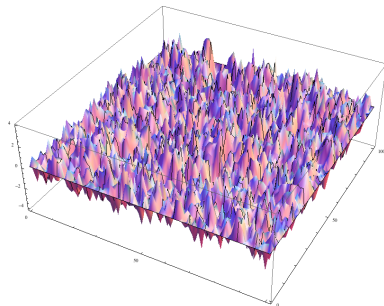
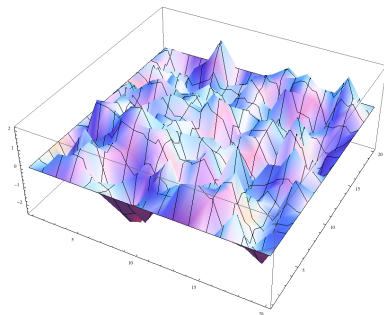
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Question: Which measure on ρ ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

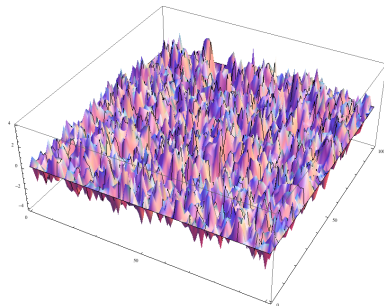
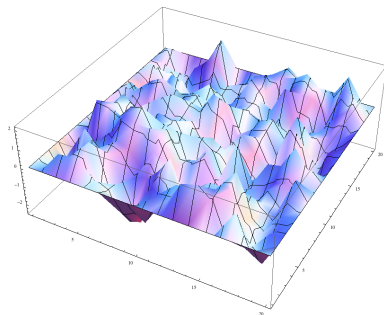
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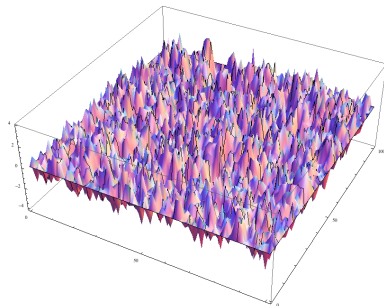
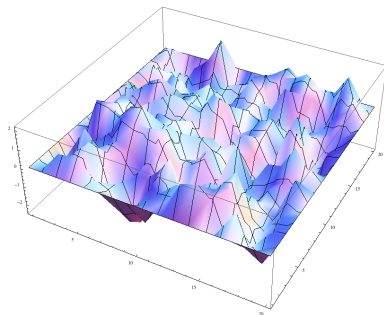


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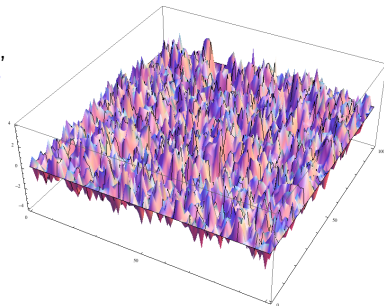
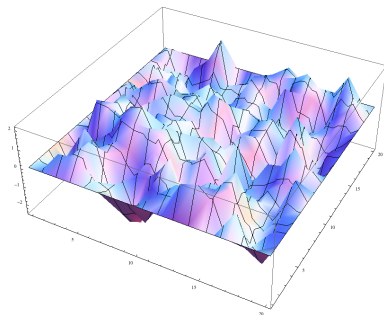
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- ▶ Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

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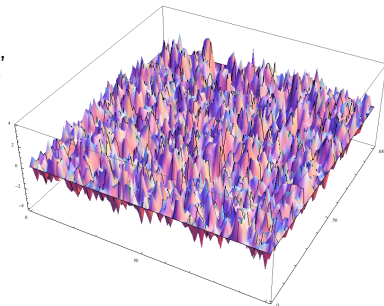
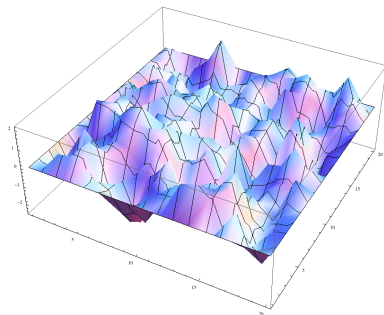
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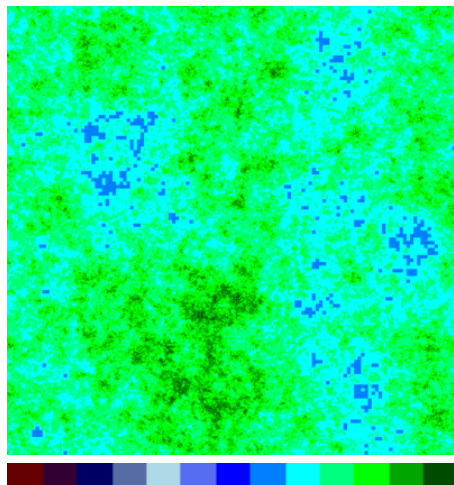
- ▶ Continuum GFF not a function — only a generalized function



Liouville quantum gravity

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where h is a GFF and $\gamma \in [0, 2)$

$$\gamma = 0.5$$

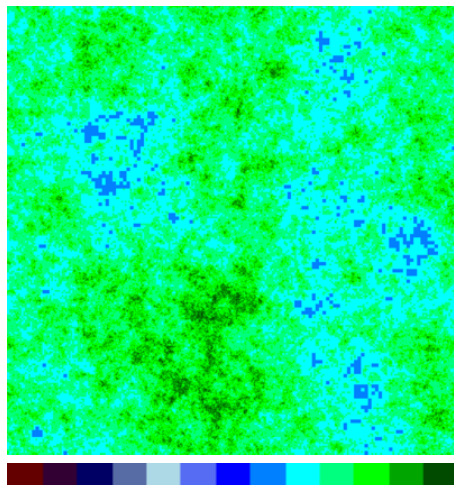


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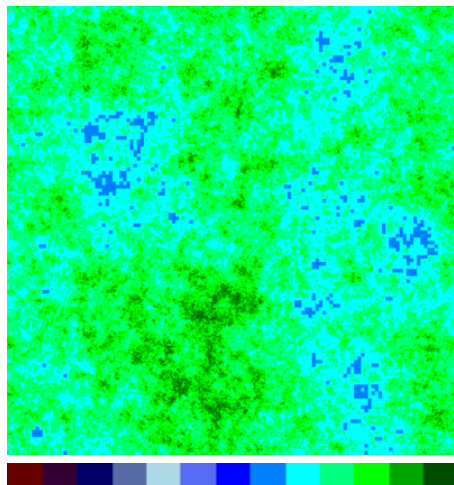


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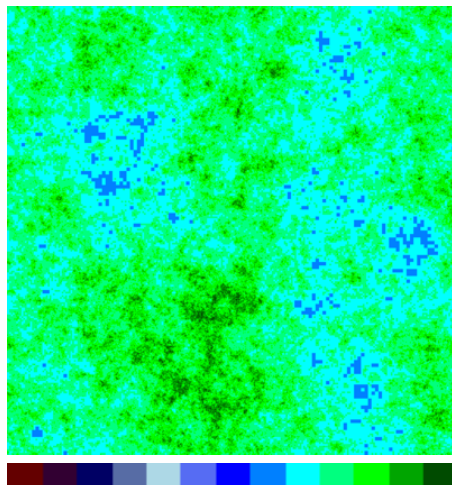


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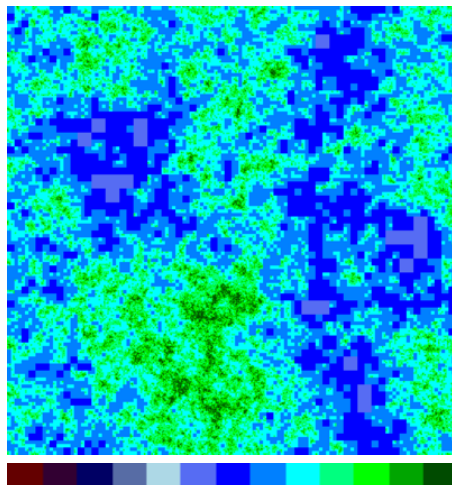


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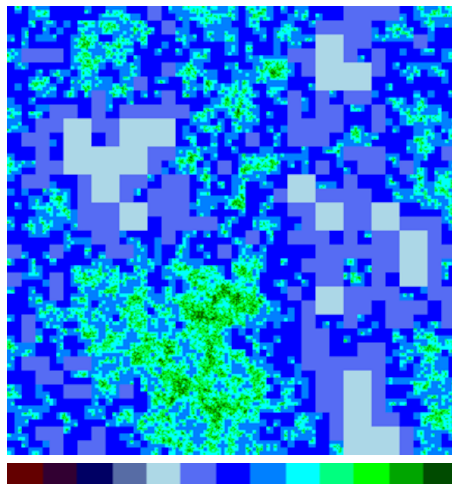


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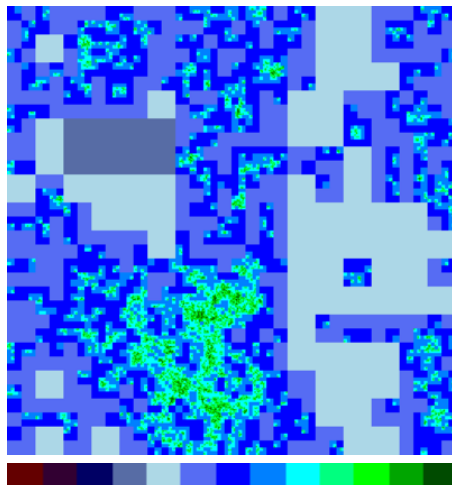


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A current direction of research

RPM - most natural random surface model for a combinatorialist

LQG - most natural random surface model for a complex analyst:

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- ▶ Relate Liouville quantum gravity to random planar maps
 - ▶ $\sqrt{8/3}$ -LQG = to the Brownian map (M.-Sheffield)
 - ▶ $\sqrt{8/3}$ -LQG + $\text{SLE}_{8/3}$ = random planar map + self-avoiding walk (Gwynne-M.)
 - ▶ $\sqrt{8/3}$ -LQG + SLE_6 = random planar map + percolation (Gwynne-M.)
 - ▶ $\sqrt{2}$ -LQG + SLE_8 = random planar map + uniform spanning tree (Sheffield, Duplantier-M.-Sheffield)

Ευχαριστώ πολύ!

